AMALGAMATION IN B-LATTICE VARIETIES

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Colin Naturman

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Henry Rose[1]

Abstract: Let $\mathbf{B}$ be a non–modular subdirectly irreducible lattice obtained from a finite Boolean lattice $\mathbf{B}'$ by splitting some element of $\mathbf{B}'$. The amalgamation class Amal($\mathcal{B}$) of the variety $\mathcal{B}$ generated by $\mathbf{B}$, is considered. It is shown that Amal($\mathcal{B}$) is closed under reduced products and there is an elementary class determined by Horn sentences. The above result is based on a characterization of Amal($\mathcal{B}$).

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INTRODUCTION

Let L' be a finite Boolean lattice, and let $d \in L'$ be neither top or bottom element of L'. By splitting d one can obtain a new lattice L which is non–modular and subdirectly irreducible (see section 1). We refer to such a lattice as a *B–lattice.* Given a B–lattice B, let $\mathcal{B}$ be the variety generated by B and let Amal($\mathcal{B}$) be the amalgamation class of $\mathcal{B}$. The principal result of this paper, Theorem 5.1, tells us that Amal($\mathcal{B}$) is closed under reduced products and therefore it is an elementary class determined by Horn sentences. The pentagon is a B–lattice and so the above theorem generalizes Theorem 3.1 of [3]. The proof of Theorem 5.1 is based on a characterization of Amal($\mathcal{B}$) (Theorem 4.2). In [9] a characterization of the finite members of the amalgamation class of the variety generated by the pentagon, is given. Theorem 4.3 generalizes this result: a finite lattice A belongs to Amal($\mathcal{B}$) if and only if A is a subdirect power of B and A does not have the three element chain as a homomorphic image.

The results of sections 0, 1, 2 and 3 are mainly technical. They are needed to prove theorems from sections 4 and 5. Since authors are currently preparing papers which investigate the amalgamation classes of varieties generated by other splitting lattices (all B–lattices are splitting) some statements in this paper are proved in a more general setting than is required for B–lattices and varieties generated by them. This mainly concerns statements 2.4 – 2.6 and statements from subsections 0.5, 0.6 and 0.9.

Final Remark: As we mentioned before, the pentagon is a B–lattice and it will be noticed that the proofs of some statements are essentially the same as in [3]. However, we have included these proofs to make this paper self–contained.

0. PRELIMINARY RESULTS, NOTATION AND TERMINOLOGY

0.1 The Pentagon and N–quotients: Let u/v be a quotient of a lattice L. We shall say that u/v is an *N–quotient* of L if for some $z \in L$, the set {u,v,z} generates a sublattice of L isomorphic to the pentagon N (see Figure 1). In this pentagon, u/v is a critical quotient. (See 0.3(i).)

0.2 n–sublattices: Given a positive integer n, we denote the chain with n elements by **n**. Let X be a sublattice of a lattice L. If $X \cong$ **n** for some positive interger n and there is a retraction of L onto X, then we say that X is an *n–sublattice* of L. For example, if L is a distributive lattice and $X \cong$ **2**, then X is a 2–sublattice of L. On the other hand, if L is non–distributive and subdirectly irreducible with critical quotient p/q (see 0.3(i)) and X = {p,q}, then X is not a 2–sublattice of L.

0.3 Congruences and Embeddings: (i) Con(A) denotes the congruence lattice of an algebra A. The smallest congruence which identifies the elements a and b of A, is denoted by Con(a,b). If A is subdirectly irreducible and Con(a,b) is the smallest non–trivial congruence on A, we call the set {a,b}, a *critical pair* of A. If A is a lattice, then we may assume without loss of generality that $a > b$. In that case we shall refer to the quotient a/b as a *critical quotient* of A.

(ii) Let A and C be algebras and let $f : A \longrightarrow C$ be a surjective homomorphism. Let $\Theta \in$ Con(A) be the kernel of f. We shall refer to Θ as a *C–congruence.* If C has some special algebraic property e.g C is subdirectly irreducibe or C is a distributive lattice etc. then we shall call Θ, subdirectly irreducible or distributive etc.

(iii) **Lemma:** (a) Let L be a lattice which does not have a diamond as a sublattice. Then $\Psi \in$ Con(L) is the smallest distributive congruence on L if and only if $\Psi = \cap_I \Theta_i$, if and only if $\Psi = \Sigma_S \mathrm{Con}(u_s/v_s)$, where $\{\Theta_i : i \in I\}$ is the set of all 2–congruences on L and { u_s/v_s :

$s \in S\}$ is the set of all N–quotients of L.

(b) Let L be a lattice and let $p > q$ in L. Let Ψ be the smallest distributive congruence on L and suppose that $(p,q) \notin \Psi$. Then if X is any sublattice of L with $\{p,q\} \subseteq X$, $\{p,q\}$ forms a 2–sublattice of X.

(iv) We use $A \leq C$ to denote that A is a subalgebra of C or equivalently, that C is an extension of A. Let $\mathcal{V}$ be a variety. We shall say that $A \in \mathcal{V}$ is *congruence extensile in* $\mathcal{V}$ if for any $C \in \mathcal{V}$ with $A \leq C$, every congruence on A can be extended to a congruence on C. An extension C of A is said to be *essential* if every non–trivial congruence on C restricts to a non–trivial congruence on A. A is said to be a *maximal irreducible in* $\mathcal{V}$, if and only if A is subdirectly irreducible and A has no subdirectly irreducible proper essential extension in $\mathcal{V}$. The class of maximal irreducibles in $\mathcal{V}$ will be denoted by $\mathcal{V}_{MI}$

(v) Let $C = \Pi_I C_i$ be a product of algebras and let $f : A \to C$ be an embedding. We shall say that f is *regular* if for any distinct $i,j \in I$ we have $\Theta_i | f[A] \neq \Theta_j | f[A]$, where Θ_i and Θ_j are the kernels of the canonical projections of C onto C_i and C_j respectively.

0.4 Filtral Congruences: Let $A = \Pi_I A_i$ be a product of algebras. A congruence Φ on A is said to be *filtral* if there is an ultrafilter $\mathcal{F}$ over I which induces Φ i.e. for all $x,y \in A$ we have $(x,y) \in \Phi$ if and only if $\{i \in I : x_i = y_i\} \in \mathcal{F}$. It is well known that if I is finite, every filtral congruence on A is an A_i congruence for some $i \in I$. Now suppose that A is congruence distributive. We shall frequently use the following fundamental result:

Jónsson's Lemma: If $C \leq A$ and Θ is a subdirectly irreducible congruence on C, then there is a filtral congruence Φ on A such that $\Phi | C \subseteq \Theta$.

The following is an important consequence of Jónsson's Lemma:

Corollary: Let $\mathcal{V}$ be a congruence distributive variety generated by a finite set of finite

algebras $\mathcal{K}$. If S is a subdirectly irreducible member of $\mathcal{V}$, then S is a homomorphic image of a subalgebra of some $A \in \mathcal{K}$.

0.5 Lattice varieties which do not have diamond and the lattices L_1, L_2 L_3, L_4, L_5, L_{11}, L_{12} as its members: Let $\mathcal{V}$ be a variety as indicated in the title of this section. (See Figures 2 and 3) and let $L \in \mathcal{V}$. The proof of the following result can be found in [10] (For the concept of weak projection the reader is referred to [5].)

Lemma A: (i) Let $p,q,z \in L$ such that $N(p/q,z)$. If u/v is a quotient of L which projects weakly onto p/q, then $N(u/v,z)$.

(ii) Assume that L is a subdirectly irreducible. Then:

(a) L has a unique critical quotient u/v.

(b) The principal filter [v) and the principal ideal (u] form distributive sublattices of L.

(c) The following are equivalent for $z \in L$:

  (1) $N(u/v,z)$

  (2) z is non–comparable with u and v.

  (3) There is a quotient p/q of L with $N(p/q,z)$.

The following proposition is concerned with the smallest distributive congruences of members of $\mathcal{V}$.

Proposition B: Let $L \in \mathcal{V}$, let Ψ be the smallest distributive congruence on L and let $\{u_s/v_s : s \in S\}$ be the set of all N–quotients of L.

(a) Let $p,q \in L$. Then $(p,q) \in \Psi$ if and only if there is a finite subset $\{x_1,\dots,x_n\}$ of L such that $p + q = x_1 > \dots > x_n = pq$ and for all $i \in \{1,\dots,n-1\}$ the quotients x_i/x_{i+1} are N–quotients in L.

(b) Let $M \leq L$. Then $\Psi | M = \Sigma_S(\mathrm{Con}(u_s/v_s) | M)$.

Proof: (a) By Lemma 0.3(a), $\Psi = \Sigma_S \mathrm{Con}(u_s/v_s)$. Note that $(p,q) \in \Psi$ if and only if there are $x_1,\ldots,x_n \in L$ with $p + q = x_1 > \ldots > x_n = pq$ such that for $i \in \{1,\ldots,n-1\}$, the quotient x_i/x_{i+1} projects weakly onto an N–quotient of L (see for example [5] page 77). The result now follows from part (i) of Lemma A.

(b) Again by Lemma 0.3(a) we have $\Psi = \Sigma_S \mathrm{Con}(u_s/v_s)$ so that $\Sigma_S(\mathrm{Con}(u_s,v_s)|M) \subseteq (\Sigma_S \mathrm{Con}(u_s,v_s))|M = \Psi|M$. Now consider $p,q \in L$ with $(p,q) \in \Psi|M$. By (i) there are $x_1,\ldots,x_n \in M$ such that $p + q = x_1 > \ldots > x_n = pq$ and for all $i \in \{1,\ldots n-1\}$ the quotient x_i/x_{i+1} is an N–quotient of M and hence of L. But then we have $(p+q,pq) \in \Sigma_S(\mathrm{Con}(u_s,v_s)|M)$ whence $(p,q) \in \Sigma_S(\mathrm{Con}(u_s,v_s)|M)$ and so the result follows.

We now turn our attention to a subdirect products of non–modular subdirectly irreducible members of $\mathcal{V}$.

Lemma C: The following are equivalent for $L \in \mathcal{V}$.

(a) L is a subdirect product of non–modular subdirectly irreducible members of $\mathcal{V}$.

(b) For every non–trivial quotient of p/q of L, there is an N–quotient s/t of L which projects weakly onto p/q.

Proof: Assume (a) and let p/q be a non–trivial quotient of L. Let Θ be a non–modular subdirectly irreducible congruence on L with $(p,q) \notin \Theta$. Let u/v be a quotient of L such that $(u/\Theta)/(v/\Theta)$ is a critical quotient in L/Θ. By Lemma 0.5 A(ii) part (c), $(u/\Theta)/(v/\Theta)$ is an N–quotient of L/Θ. Since the pentagon is a projective lattice we may assume that $N(u/v,z)$ for some $z \in L$. Now $(u/\Theta)/(v/\Theta)$ projects weakly onto $(p+q/\Theta)/(pq/\Theta)$ in L/Θ. Thus there is a quotient s/t in L which projects weakly onto both u/v and p/q (see [12]). Since $N(u/v,z)$, it follows from Lemma 0.5 A(i) that $N(s/t,z)$. Thus (a) implies (b). Now assume (b). Let $\{p_i/q_i : i \in I\}$ be the set of all non–trivial quotients of L. By assumption, for every $i \in I$ there is an N–quotient s_i/t_i which projects weakly onto p_i/q_i. For each $i \in I$, let Θ_i be a subdirectly irreducible congruence on L with $(p_i,q_i) \notin \Theta_i$. Then

for all $i \in I$, $(s_i,t_i) \notin \Theta_i$, and so Θ_i is a non–modular subdirectly irreducible congruence on L. Note that $\cap_I \Theta_i$ is the trivial congruence on L and the so (a) follows. Thus (b) implies (a).

Lemma D: Assume that $\mathcal{V}$ is finitely generated. Let $L \in \mathcal{V}$ and let $C = \Pi_I C_i$ where for all $i \in I$, $C_i \in \mathcal{V}_{MI}$. Suppose that there is an embedding $f : L \longrightarrow C$ with the following property: For every homomorphism $g : L \longrightarrow M$ where $M \in \mathcal{V}_{MI}$ and for which g[L] contains a unique critical quotient of M, there is a homomorphism $h : C \longrightarrow M$ with $h \circ f = g$. Then for some $i \in I$, h[C] is an image of C_i.

Proof: Since g[L] contains a unique critical quotient of M and $g[L] \leq h[C]$, the lattice h[C] must be directly indecomposable. Since $\mathcal{V}$ is finitely generated we see by the Corollary 0.4 that M and hence h[C] is finite and so the result follows by Theorem 7 of [2].

0.6 Reduced Products: In this subsection we assume that $\mathcal{V}$ is a lattice variety as in section 0.5. Let $\mathcal{X}$ be the class of lattices in $\mathcal{V}$ which are subdirect products of non–modular subdirectly irreducible members of $\mathcal{V}$. We are going to show that under certain assumptions, the class $\mathcal{X}$ is closed under reduced products.

Notation: Given quotients s/t and p/q of a lattice L, we shall write $\langle s/t,p/q\rangle_n$ if n is the smallest number of steps for which s/t projects weakly onto p/q.

Definition: (i) Let S be a subdirectly irreducible member of $\mathcal{V}$ and let c/a be a (unique) critical quotient of S. Let r be an integer. We say that S has radius r if $r = \max\{n :$ there is a quotient p/q of S with $\langle c/a,p/q\rangle_n\}$.

(ii) A class $\mathcal{S}$ of subdirectly irreducible members of $\mathcal{V}$ is said to be k–*restricted* if k is the maximum radius of members of $\mathcal{S}$.

From now on $\mathcal{S}$ will denote the set of all non–modular subdirectly irreducible members of $\mathcal{V}$ and we will be assuming that $\mathcal{S}$ is k–restricted for some fixed integer k. Note that $\mathcal{X}$ is the class of subdirect products of members of $\mathcal{S}$. Consider an $L \in \mathcal{X}$. If s/t and p/q are quotients of L with $\langle s/t,p/q\rangle_n$, then $n \le k + 2$ (see [12]). Thus under our assumption, Lemma 0.5 C can be reformulated as follows:

Lemma: Let $L \in \mathcal{V}$. Then $L \in \mathcal{X}$ if and only if for every non–trivial quotient p/q, there is an N–quotient s/t such that $\langle s/t,p/q\rangle_n$ and $n \le k + 2$.

Proposition: The class $\mathcal{X}$ is closed under reduced products.

Proof: Let $L = \Pi_I L_i$ where for all $i \in I$, $L_i \in \mathcal{X}$. Let $\mathcal{F}$ be a filter over I and let Θ be the congruence on L induced by $\mathcal{F}$. Let $p,q \in L$ be such that $(p/\Theta)/(q/\Theta)$ be a non–trivial quotient of L/Θ. Put $Y = \{i \in I : p_i > q_i\}$. Then $Y \in \mathcal{F}$. Since $L_i \in \mathcal{X}$ for all $i \in I$, we see by the preceding lemma that for any $i \in Y$ there is an N–quotient s_i/t_i of L_i such that $\langle s_i/t_i,p_i/q_i\rangle_n$ with $n \le k + 2$. For all $i \in I - Y$ let $s_i \in L_i$ and let $t_i = s_i$. Putting $s = (s_i)_I$ and $t = (t_i)_I$ we obtain an N–quotient s/t in L with $\langle (s/\Theta)/(t/\Theta) , (p/\Theta)/(q/\Theta) \rangle_n$ in L/Θ. The result follows now from the preceding lemma.

0.7 The Amalgamation Class of a Variety: By a *diagram* in a variety $\mathcal{V}$, we mean a quintuple (A,f,C,g,D) where $f : A \longrightarrow C$ and $g : A \longrightarrow D$ are embeddings with $A,C,D \in \mathcal{V}$. By an *amalgam* in $\mathcal{V}$ of this diagram, we mean a triple (E,f',g') where $f' : C \longrightarrow E$ and $g' : D \longrightarrow E$ are embeddings with $E \in \mathcal{V}$ and $f \circ f' = g \circ g'$. If such an amalgam exists, we say that the diagram can be amalgamated in $\mathcal{V}$. An algebra $A \in \mathcal{V}$ is called an *amalgamation base* for $\mathcal{V}$ if every diagram {a,f,C,g,D) in $\mathcal{V}$ can be amalgamated in $\mathcal{V}$. The class of all amalgamation bases for $\mathcal{V}$ is called the *amalgamation class* of $\mathcal{V}$ and is denoted by Amal($\mathcal{V}$). If $\mathcal{V}$ is non–trivial then Amal($\mathcal{V}$) will be a proper class. (See [13].)

0.8 Absolute Retracts: Let $\mathcal{V}$ be a variety. An algebra $A \in \mathcal{V}$ is said to be an *absolute retract* in $\mathcal{V}$ if for any embeddings $f : A \longrightarrow C$ with $C \in \mathcal{V}$, there is a (surjective) homomorphism $g : C \longrightarrow A$ such that $g \circ f$ is the identity map on A. Absolute retracts of a variety $\mathcal{V}$ belong to Amal($\mathcal{V}$) (see [1] Proposition 3.2). absolute retracts in a variety $\mathcal{V}$ have no proper essential extensions in $\mathcal{V}$. It is well known that $\mathcal{V}$ is residually small if and only if every member of $\mathcal{V}$ is essentially embeddable into an absolute retract in $\mathcal{V}$ (see [11]). If A is an essential extension of a subdirectly irreducible, then A is itself subdirectly irreducible. The subdirectly irreducible and so if $\mathcal{V}$ is residually small, every subdirectly irreducible member of $\mathcal{V}$ is embeddable in a subdirectly irreducible absolute retract in $\mathcal{V}$ and moreover, the class of subdirectly irreducible absolute retracts in $\mathcal{V}$ is precisely $\mathcal{V}_{MI}$ (see 0.3(iv)). For example, if L is any one of the lattices in Figures 1, 2 or 3, and $\mathcal{V}$ is generated by L, then $\mathcal{V}_{MI} = \{L\}$.

0.9 Amalgamation in residually small congruence distributive varieties whose members have one element subalgebras: Let $\mathcal{V}$ be a variety as indicated in the title. The proof of the following result can be found in [7].

Theorem A: (i) $A \in$ Amal($\mathcal{V}$) if and only if for any embedding $f : A \longrightarrow C$ with $C \in \mathcal{V}$, and for any homomorphism $g : A \longrightarrow M$ with $M \in \mathcal{V}_{MI}$, there is a homomorphism $h : C \longrightarrow M$ such that $h \circ f = g$.

(ii) A product of absolute retracts of $\mathcal{V}$ is an absolute retract and therefore belongs to Amal($\mathcal{V}$).

Remark B: (i) We note that residual smallness is not required in part (ii) of Theorem A.

(ii) Part (i) of Theorem A remains valid if we only consider the case when A and g[A] are non–trivial.

(iii) Let A be as in the first part of the Theorem A. Then A is congruence extensile.

Example C: Let L be one of the lattices in Figure 1, 2 or 3, and let $\mathcal{V}$ be the variety generated by L. Then any direct power of L is an absolute retract in $\mathcal{V}$ and therefore belongs to Amal($\mathcal{V}$).

The following result is Corollary 1.1.3 of [3].

Lemma D: Assume that $\mathcal{V}$ is finitely generated and that for any distinct $D,E \in \mathcal{V}_{MI}$, D is not an image of a subalgebra of E. Let A be a subdirect product of members of $\mathcal{V}_{MI}$. Then $A \in$ Amal($\mathcal{V}$) if and only if for any regular subdirect representation $f : A \longrightarrow C$ where C is a product of members of $\mathcal{V}_{MI}$, and any homomorphism $g : A \longrightarrow M$ where $M \in \mathcal{V}_{MI}$, there is a homomorphism $h : C \longrightarrow M$ such that $g = h \circ f$.

The assumptions which are required to prove the preceding lemma can be dropped if we assume that A is congruence extensile.

Lemma E: Let $\mathcal{V}$ be as in Theorem 0.9 A, and assume that A is a subdirect product of members of $\mathcal{V}_{MI}$. Then $A \in$ Amal($\mathcal{V}$) if and only if the following two conditions hold:

(i) A is congruence extensile.

(ii) If C is a product of members of $\mathcal{V}_{MI}$, $f : A \longrightarrow C$ is a regular subdirect representation, and $g : A \longrightarrow M$ is a homomorphism with $M \in \mathcal{V}_{MI}$, then there is a homomorphism $h : C \longrightarrow M$ such that $h \circ f = g$.

Proof: If $A \in$ Amal($\mathcal{V}$), then (i) and (ii) hold by Theorem 0.9 A(i). For the converse assume that (i) and (ii) hold. Let $C' = \Pi_I S_i$ where for all $i \in I$, $S_i \in \mathcal{V}_{MI}$ and let $f' : A \longrightarrow C$ be a subdirect representation. Since A is congruence extensile, it follows from Lemma 3.10 in [1], that the following is sufficient for $A \in$ Amal($\mathcal{V}$):

(a) $C' \in$ Amal($\mathcal{V}$)

(b) If $g' : A \longrightarrow M$ is a homomorphism with $M \in \mathcal{V}_{MI}$ then there is a homomorphism $h' : C' \longrightarrow M$ such that $h' \circ f' = g'$.

Now (a) follows from Theorem 0.9 A(ii). We complete the proof by showing that (ii) implies (b). So let $g' : A \longrightarrow M$ be a homomorphism with $M \in \mathcal{V}_{MI}$. Now the subdirect representation $f' : A \longrightarrow C'$ induces a regular representation $f : A \longrightarrow C$ where $C = \Pi_J S_i$ for some $J \subseteq I$. For each $i \in J$ we have the homomorphism $\pi_i \circ f' : A \longrightarrow S_i$ where $\pi_i : C' \longrightarrow S_i$ is the ith projection, and so by (ii) there is a homomorphism $\kappa_i : C \longrightarrow S_i$ such that $\kappa_i \circ f = \pi_i \circ f'$. For $i \notin J$ let $\kappa_i : C \longrightarrow S_i$ be the ith projection. Now let $k : C \longrightarrow C'$ be the product homomorphism induced by the homomorphisms κ_i, $i \in I$. Then k is an embedding and $k \circ f = f'$. By (ii) there is a homomorphism $h : C \longrightarrow M$ such that $h \circ f = g'$. By Theorem 0.9 A(ii), $C \in$ Amal($\mathcal{V}$). By Theorem 0.9 A(i) there is a homomorphism $h' : C' \longrightarrow M$ such that $h' \circ k = h$. It follows that $h' \circ f' = g'$ as required.

Lemma F: Let $M \in \mathcal{V}_{MI}$ and let P be a retract of M. Let $A \in$ Amal($\mathcal{V}$) and let $f : A \longrightarrow C$ be an embedding with $C \in \mathcal{V}$. Then every P–congruence on A can be extended to a P–congruence on C.

Proof: Let Θ be a P–congruence on A. There is a homomorphism $g : A \longrightarrow M$ with $\Theta =$ ker(g). By Theorem 0.9 A(i) there is an $h : C \longrightarrow M$ such that $h \circ f = g$. Let $k : M \longrightarrow P$ be a retraction and let $\Psi = \ker(k \circ h)$. Then Ψ is a P–congruence on C which extends Θ.

Lemma G: Let S be a subdirectly irreducible member of $\mathcal{V}$ and let $M \in \mathcal{V}_{MI}$. Suppose further that there are two embeddings $f,g : S \longrightarrow M$ such that f is essential and g is not. Then $S \notin$ Amal($\mathcal{V}$).

Proof: Suppose that $S \in$ Amal($\mathcal{V}$). Then by Theorem 0.9 A(i) there is an $h : M \longrightarrow M$ with $h \circ f = g$. Let $\{u,v\}$ be a critical pair of S. Then $\{f(u),f(v)\}$ is a critical pair of M since f is essential (see 0.8). Thus h must be an automorphism of M. But then $\{g(u),g(v)\}$ is a critical pair of M since $g = h \circ f$, a contradiction since g is not essential.

We end this section with two corollaries concerning the case when $\mathcal{V}$ is a residually small lattice variety.

Corollary H: Suppose that $\mathcal{V}$ is a residually small lattice variety and assume that there is a non–distributive lattice $M \in \mathcal{V}_{MI}$ which has a $\mathbf{2}$–sublattice. Then $\mathbf{2} \notin \mathrm{Amal}(\mathcal{V})$.

Proof: Let u/v be a critical quotient of M and let $\{s,t\}$ form a $\mathbf{2}$–sublattice of M with $s < t$. Then $\{u,v\} \neq \{s,t\}$ (see 0.2). Considering embeddings $f,g : \mathbf{2} \longrightarrow M$ with $f[\mathbf{2}] = \{u,v\}$ and $g[\mathbf{2}] = \{s,t\}$ we see that the result follows from the preceding Lemma.

Theorem 6.2 in [9] tells us that if $\mathcal{V}$ is the lattice variety generated by the pentagon, then every non–trivial member of $\mathrm{Amal}(\mathcal{V})$ is a subdirect product of pentagons. The proof goes as follows: Let $A \in \mathrm{Amal}(\mathcal{V})$ and let $k : A \longrightarrow \Pi_I(A/\Theta_i)$ be a subdirect representation. Put $J = \{i \in I : A/\Theta_i \cong N\}$ and $S = \{i \in I : A/\Theta_i \cong \mathbf{2}\}$. If S is not empty then there are $x,y \in A$ with $x > y$ and $(x,y) \notin \cap_J \Theta_i$. Let $f : \mathbf{2} \longrightarrow A$ be the embedding with $f[\mathbf{2}] = \{x,y\}$, and let $g : \mathbf{2} \longrightarrow C$ be any embedding with $C \in \mathcal{V}$. Using the assumption that S is not empty it is shown that the diagram $(\mathbf{2},f,A,g,C)$ can be amalgamated in $\mathcal{V}$ and therefore $\mathbf{2}$ must belong to $\mathrm{Amal}(\mathcal{V})$ (see [6]). The argument remains exactly the same if we assume that A is a member of a non–distributive lattice variety and we put $J = \{i \in I : \Theta_i$ is a non–distributive subdirectly irreducible congruence on $A\}$. Combining this with the preceding corollary we have:

Corollary I: Suppose that $\mathcal{V}$ is a residually small lattice variety and assume that there is a non–distributive lattice $M \in \mathcal{V}_{MI}$ which has a $\mathbf{2}$–sublattice. Then every member of $\mathrm{Amal}(\mathcal{V})$ is a subdirect product of non–distributive subdirectly irreducible members of $\mathcal{V}$.

1. THE CLASS $\mathcal{K}$

1.1 Members of $\mathcal{K}$: Let d be an element of a lattice L'. By splitting d one can obtain a new lattice L. More precisely we put $L = (L' - \{d\}) \cup \{a,c\}$ where $a = (d,0)$ and $c = (d,1)$. L is then the lattice with underlying set L and with partial order defined as follows: Given $x,y \in L$, xe have $x \leq y$ in L if and only if one of the following holds:

(a) $x,y \in L' - \{d\}$ and $x \leq y$ in L'

(b) $x \in \{a,c\}$, $y \in L' - \{d\}$ and $d \leq y$ in L'

(c) $x \in L' - \{d\}$, $y \in \{a,c\}$ and $x \leq d$ in L'

(d) $(x,y) \in \{ (a,a), (a,c), (c,c) \}$.

Now let $\mathcal{K}$ be the class of all non–modular subdirectly irreducible lattices L obtained by splitting some element of a finite distributive lattice L'. For example the pentagon and the lattices L^{k}_{13}, L_{14} and L^{q}_{15} are in $\mathcal{K}$ (see Figure 3). A detailed study of the class $\mathcal{K}$ can be found in [10] Chapter 5. In particular, the proofs of the following two results can be found there.

Proposition 1.1.1: Let $L \in \mathcal{K}$ be obtained by splitting the element d of a finite distributive lattice L' and let $a,c \in L$ be as above. Then the following statements all hold:

(a) Either $L \cong N$ or for some $i \in \{13,14,15\}$, L_i is embeddable in L (see Figure 3).

(b) Every non–distributive subdirectly irreducible member of the variety generated by L belongs to $\mathcal{K}$.

(c) c/a is the unique N–quotient of L and it is a critical quotient with $L/\mathrm{Con}(a,c) \cong L'$.

(d) The principal filter $[d)$ and the principal ideal $(d]$ of L' form non–trivial Boolean sublattices of L'.

Theorem 1.1.2 (Characterization Theorem)

Let L' be a finite distributive lattice and let $d \in L'$ be neither the top nor bottom element of L$'$. Let L be obtained from L$'$ by splitting d. Then L $\in \mathcal{K}$ (i.e. L is non–modular and subdirectly irreducible) if and only if the following all hold:

(a) Every cover of d in L$'$ is join reducible.

(b) Every dual cover of d in L$'$ is meet reducible.

(c) Every prime quotient in L$'$ projects onto a prime quotient p/q with either p = d or q = d.

From this Characterization Theorem we obtain the following:

Corollary 1.1.3: If L$'$ is a finite Boolean lattice and d $\in$ L$'$ is neither the top or bottome element of L$'$, then L $\in \mathcal{K}$.

Proof: Consider a cover x of d in L$'$. Since L$'$ is Boolean and d is not the bottom element of L$'$, the principal ideal (x] forms a Boolean sublattice of L$'$ with at least four elements. Thus x must be join reducible. Dually we see that if c is a dual cover of d in L$'$, it is meet reducible. Thus conditions (a) and (b) of Theorem 1.1.2 hold. To show that condition (3) also holds, consider a prime quotient x/y of L$'$. There is an atom z of L$'$ such that z/0 transposes up onto x/y, where 0 is the bottom element of L$'$. Then either z $\leq$ d or z is non–comparable with d. If z $\leq$ d, then there is a dual cover q of d such that d/q transposes down onto z/0 whence (c) holds. If z is non–comparable with d then the prime quotient (d + z)/d transposes down onto z/0 and again (c) holds.

Remark 1.1.4: We shall refer to the members of $\mathcal{K}$ obtained from finite Boolean lattices as *B–lattices*. It will follow from Corollary 1.1.6 below, that every member of $\mathcal{K}$ is either a B–lattice or isomorphic to L^n_{15} for some non–negative integer n (see Figure 3). By Proposition 1.1.1(a), every member of $\mathcal{K}$ which is not isomorphic to the pentagon has a sublattice isomorphic to L_{13}, L_{14}, or L_{15}. It is shown below that if L $\in \mathcal{K}$ and L_{13} or L_{14} is embeddable in L, then L is a B–lattice. On the other hand, if L $\in \mathcal{K}$ but L_{13} and L_{14} are not embeddable in L and L is not isomorphic to N, then L is in fact isomorphic to L^n_{15} for some non–negative integer n.

Lemma 1.1.5: Let L $\in \mathcal{K}$ and assume that either L_{13} or L_{14} is embeddable in L. Then L is a B–lattice.

Proof: Consider the case when L_{13} is embeddable in L. Let k = max {m : m is a non–negative integer with L^m_{12} embeddable in L}. Let a,c,u,b_1,...,b_k $\in$ L^k_{13} be as in Figure 3. Without loss of generality we may consider $L^k_{13} \leq$ L with b_1,...,b_k covering u in L. Since L $\in \mathcal{K}$ there is a finite distributive lattice L$'$ such that L is obtained from L$'$ by splitting some d $\in$ L$'$. By Proposition 1.1.1(c) we see that we must have a = (d,0) and c = (d,1). By Proposition 1.1.1(d), the principal ideal (d] of L$'$ forms a Boolean sublattice. Let a_1,...,a_n be the atoms of (d]. We claim that u is the bottom element of L$'$. Suppose that this is not the case. Then there is a dual cover v of u in L$'$. By Theorem 1.1.2 (c) the prime quotient u/v of L$'$ projects onto a prime quotient p/q of L$'$ with either p = d or q = d. Since L$'$ is distributive, there is a prime quotient x/y of L$'$ such that p/q transposes onto x/y and x/y transposes onto u/v. Since d $\geq$ u > v, this implies that q $\neq$ d and so p = d. By Proposition 1.1.1(d) the principal filter [d) is forms a Boolean sublattice of L$'$. Thus the quotient d/v forms a Boolean sublattice as well. Since d > u >v, d/v must have at least four elements. Thus v has a cover z with z $\neq$ u. Note that u = db_i for any i $\in$ {1,...,k} and so z is non–comparable with b_1,...,b_k. For all i $\in$ {1,...,k} we must have u + z $\neq$ b_i or else N(b_i/u,z) contradicting the distributivity of L$'$. Also u + z must cover u. Thus the subset {a_1,...,a_n,b_1,...,b_k,u + z} generates a Boolean sublattice of L$'$ with at least k + 2 atoms. But this implies that L^{k+2}_{13} is embeddable L contradicting the maximality of k. Thus u is the bottom element of L$'$ as claimed. Now put t = d + b_1 + ... + b_k. Note that the principal ideal (t] forms a Boolean sublattice of L$'$ with atoms a_1,...,a_n,b_1,...,b_k. Suppose that t is not the top element of L$'$. Then d has a cover w in L$'$ with w $\neq$ b_i + d for all i $\in$ {1,...,k}. But by Theorem 1.1.2(a), there are r,s < w in L$'$ with r + s = w. Since w covers d we have $r \nleq d$ and $s \nleq d$. Also we cannot have both $r \leq d$ and $s \leq d$ and so without loss of generality we may assume that s is non–comparable with d. Then $d + s = w$ and so it follows that s is also non–comparable with $b_1,\dots,b_k$. Thus we see that the set $\{d,b_1,\dots,b_k,s\}$ generates a Boolean sublattice of L' with at least $k + 2$ atoms. But then L_{13}^{k+2} is embeddable L contradicting the maximality of k. Thus t is the top element of L' and so L' is Boolean whence L is a B–lattice. The result for the case when L_{14} is embeddable in L follows dually.

Corollary 1.1.6: Every member of $\mathcal{K}$ is either a B–lattice or isomorphic to L_{15}^{n}, for some non–negative integer n.

Proof: Let $L \in \mathcal{K}$. By Proposition 1.1.1(a), either $L \cong N$ or for some $i \in \{13,14,15\}$, L_i is embeddable in L. Obviously L is a B–lattice if $L \cong N$ and by Lemma 1.1.5, L is a B–lattice if L_{13} or L_{14} is embeddable in L. Moreover by Proposition 5.3.3 of [10] we see that if L_{13} and L_{14} are not embeddable in L and L is not isomorphic to N, then then $L \cong L_{15}^{n}$ and the proof is complete.

1.2 B–Lattices and B–Varieties: For the rest of this paper let B be an arbitrary B–lattice and let c/a be the unique N–quotient of B which is also a critical quotient (see Proposition 1.1.1(c)). Let B' be the finite Boolean lattice from which B is obtained. Let $\mathcal{B}$ denote the variety generated by B. The following proposition contains the basic facts about B–lattices.

Proposition 1.2.1: (i) Let $p,q \in B$ with $p > q$ and $\{p,q\} \neq \{a,c\}$. If X is any sublattice of B and $\{p,q\} \subseteq X$, then $\{p,q\}$ forms a 2–sublattice of X.

(ii) Let A be a direct product of B–lattices and assume that only finitely many factors of A are non–isomorphic. Then every finite distributive image of A is Boolean.

Proof: Let X be a sublattice of B with $\{p,q\} \subseteq X$. Let $\Theta = \mathrm{Con}(a,c)|X$. Since $\{p,q\} \neq \{a,c\}$

we see that $p/\Theta > q/\Theta$ in X/Θ. The result follows now from 0.3(iii) Lemma (b). (ii) Since every distributive image of a B–lattice is a finite Boolean lattice, the result follows from Theorem 7 of [2].

To prove Corollary 1.2.3 below we need a simple fact concerning finite Boolean lattices. The following result is folkloric; we include a proof for completeness.

For a set U we will denote the Boolean lattice of subsets of U by $\mathcal{P}(U)$.

Lemma 1.2.2: Let Y be a finite set and let X be a proper subset of Y. Then for any non–empty proper subset A of X there are two distinct embeddings $f,g : \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$ with $f(A) = g(A)$ such that there is no automorphism h of $\mathcal{P}(Y)$ with $f = h \circ g$.

Proof: Let $f : \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$ be the inclusion map and let $g : \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$ be defined as follows: for all $B \subseteq A$ put $g(B) = B$ and for all $B \subseteq X - A$ put $g(B) = (Y - X) \cup B$. Then f and g are distinct lattice embeddings with $f(A) = g(A)$. Now any automorphism of $\mathcal{P}(Y)$ preserves heights of elements of $\mathcal{P}(Y)$, however,$f(X - A) = X - A$ and $g(X - A) = (Y - X) \cup (X - A)$ have distinct heights.

We will say that a lattice A is *automorphic* in a lattice C, if for any two embeddings $f,g : A \longrightarrow C$ there is an automorphism h of C with $f = h \circ g$.

Corollary 1.2.3: Let $A \in \mathcal{B}$ and assume that A is B–lattice which is not isomorphic to B. Then A is not automorphic in B.

Proof: Let A' be the finite Boolean lattice from which A is obtained by splitting some element $e \in A'$ and let $x = (e,0)$ and $y = (e,1)$. Note that there is a bijection between the set of embeddings of A into B and the set of embeddings of A' into B' which send e to d. This bijection is given explicitly as follows: If $f : A \longrightarrow B$ is an embedding then the corresponding embedding $f' : A' \longrightarrow B'$ is given by $f'(e) = d$ and $f'(z) = f(z)$ for $z \neq e$. Thus we see that A is automorphic in B if and only if for any two embeddings $f,g : A' \longrightarrow B'$ with $f(e) = g(e) = d$ there is an automorphism h of B′ with $f = h \circ g$. Since A is not isomorphic to B and A is a B–lattice we have $|A'| < |B'|$. The result follows now from the preceding lemma and the fact that every finite Boolean lattice is isomomorphic to a lattice of the form $\mathcal{P}(X)$ for some finite set X.

Our next lemma shows that every subdirectly irreducible member of $\mathcal{B}$ is essentially embeddable into B.

Lemma 1.2.4: Up to isomorphism, B is the only member of $\mathcal{B}_{MI}$.

Proof: We have to show that every subdirectly irreducible member of $\mathcal{B}$ is essentially embeddable into B (see 0.8). So let S be a subdirectly irreducible member of $\mathcal{B}$. If $S \cong 2$ then trivially S is essentially embeddable in B. Consider the case when S is not isomorphic to 2. By Proposition 1.1.1(i), $S \in \mathcal{K}$. By Corollary 1.1.6 we see that S is either a B–lattice or $S \cong L^n_{15}$ for some non–negative integer n. If S is a B–lattice and S is not isomorphic to B and so from the proof of Corollary 1.2.3 we see that S is essentially embeddable in B. Thus we are left with the case when $S \cong L^n_{15}$ for some non–negative integer n. For this case we note that by Corollary 0.4, S is an image of a sublattice X of B. Note that $a,c \in X$ and so X is obtained from a sublattice X′ of B′ by splitting the element d which is contained in X′. Let S′ denote the quotient of S under the smallest congruence which collapses its unique N–cquotient. The surjective homomorphism from X to S induces a surjective homomorphism $f : X' \longrightarrow S'$. Let $e = f(d)$. Since $S \in \mathcal{K}$, the principal filter $[e)$ and the principal ideal $(e]$ form Boolean sublattices of S′ (see Proposition 1.1.1(ii)). Also, $X' \cap [d)$ and $X' \cap (d]$ form sublattices of X′. $[e)$ and $(e]$ are images of $X' \cap [d)$ and $X' \cap (d]$ respectively, under f. Since finite Boolean lattices are projective in the variety of distributive lattices it follows that $[e)$ and $(e]$ are embeddable in $X' \cap [d)$ and $X' \cap (d]$

respectively. Thus d has at least as many covers in B′ as e has in S′ and similarly for dual covers. Since $S \cong L^n_{15}$ we can choose n covers $u_1,\dots,u_n$ of d in B′ and n dual covers $v_1,\dots,v_n$ of d in B′, such that for each $i \in \{1,\dots,n\}$ we have $u_i > d > v_i$. Let $b_1,\dots,b_n \in B'$ be such that $d + b_i = u_i$ and $db_i = v_i$. Then for each $i \in \{1,\dots,n\}$ we have $N(c/a,b_i)$ in B, so that the set $\{a,c,b_1,\dots,b_n\}$ generates a sublattice of B isomorphic to L^n_{15}.

Note that the preceding lemma shows that every member of $\mathcal{B}$ is embeddable in a direct power of B.

Lemma 1.2.5: Let $A \in \mathcal{B}$ and assume that 3 is not an image of A. Let $f : A \longrightarrow M$ be an embedding with M a direct power of B, and let $g : A \longrightarrow B$ be a homomorphism such that g[A] is a non–modular subdirectly irreducible lattice. Then there is a surjective homomorphism $h : M \longrightarrow B$ with $h \circ f = g$.

Proof: Without loss of generality we may assume that $A \leq M$ and that $f : A \longrightarrow M$ is the inclusion map. Since g[A] is subdirectly irreducible, by Jónsson's Lemma there is a filtral congruence Φ on M such that $\Phi | A \subseteq \ker(g)$. Note that Φ must be a B–congruence on M (see Corollary 0.4). Let $k : M \longrightarrow B$ be a surjective homomorphism with $\Phi = \ker(k)$. Then $\ker(k | A) \subseteq \ker(g)$ and so there is a surjective homomorphism $s : k[A] \longrightarrow g[A]$. Thus k[A] is non–modular and so it contains the unique N–quotient c/a of B. Thus k[A] is obtained from a sublattice X of B with $d \in X$, by splitting d. Since 3 is not an image of A and $X \cong k[A]/\mathrm{Con}(a,c)$, X must be a Boolean lattice. Thus k[A] is B–lattice. It follows that s must be an isomorphism. Let $h = j \circ s \circ k : M \longrightarrow B$ where $j : g[A] \longrightarrow B$ is the inclusion map. Then we have $h \circ f = g$.

2. EXTENSIONS OF 2–CONGRUENCES

It follows from Proposition 1.2.1 (i) that every B–lattice has a 2–sublattice. By applying

Lemma 0.9 F with $\mathcal{V} = \mathcal{B}$ we have the following:

Lemma 2.1: Let $A \in \mathrm{Amal}(\mathcal{B})$ and let $C \in \mathcal{B}$ with $A \leq C$. Then every 2–congruence on A can be extended to a 2–congruence on C.

Our next lemma is the simple consequence of distributivity. We shall state it without proof.

Lemma 2.2: (i) Let A and C be distributive lattices with $A \leq C$. Then every 2–congruence on A can be extend to a 2–congruence on C.
(ii) Let A and C be arbitrary lattices and let $f : A \longrightarrow C$ be an embedding. Suppose further that every 2–congruence on f[A] can be extended to a 2–congruence on C. Let $g : A \longrightarrow X$ be a surjective homomorphism where X is a finite Boolean lattice. Then there is a surjective homomorphism $h : C \longrightarrow X$ with $h \circ f = g$.

Lemma 2.3: Let A and C be arbitrary lattices with $A \leq C$. Let Ψ_A and Ψ_C be the smallest distributive congruences on A and C respectively. Then every 2–congruence on A can be extended to a 2–congruence on C, if and only if $\Psi_A = \Psi_C|A$.
Proof: Let X_C be the set of all 2–congruence on C and let $X_A = \{ \Phi \in X_C : \Phi|A$ is a 2–congruence on A $\}$. Assume that every 2–congruence on A can be extended to a 2–congruence on C. We show that $\Psi_A = \Psi_C|A$. Let $\Theta = \cap X_A$. Then Θ is a distributive congruence on C which extends Ψ_A. Since $X_A \subseteq X_C$ we have $\Psi_C = \cap X_C \subseteq \Theta$ and so $\Psi_C|A \subseteq \Theta|A = \Psi_A$. But he $\Psi_C|A$ is a distributive congruence on A and so $\Psi_A \subseteq \Psi_C|A$ whence $\Psi_A = \Psi_C|A$. For the converse assume that $\Psi_A = \Psi_C|A$. Then the map $f : A/\Psi_A \longrightarrow C/\Psi_C$ given by $f(a/\Psi_A) = a/\Psi_C$ is a well–defined embedding. Consider a 2–congruence Θ on A. Then $\Psi_A \subseteq \Theta$ and so we have a 2–congruence Θ/Ψ_A on A/Ψ_A. Let Ω be the congruence on $f[A/\Psi_A]$ corresponding to Θ. Now $f[A/\Psi_A]$ and C/Ψ_C are distributive lattices with

$f[A/\Psi_A] \leq C/\Psi_C$ and Ω is a 2–congruence on $f[A/\Psi_A]$. Thus so by Lemma 2.2(i) we see that there is a congruence Φ on C such that Φ/Ψ_C is a 2–congruence on C/Ψ_C extending Ω. But then Φ is a 2–congruence on C extending Θ and the result follows.

Lemma 2.4: (c.f. Lemma 1.2.3 in [3].) Let $\mathcal{V}$ be a lattice variety which does not contain the diamond or the lattices L_1, L_2, L_3, L_4, L_{11} or L_{12} (see 0.5 and Figures 2 & 3). Let $C \in \mathcal{V}$ and let $A \leq C$. Then the following are equivalent:

(i) Every 2–congruence on A can be extended to a 2–congruence on C.

(ii) For any distributive congruence Θ on A, and any N–quotient u/v of C, if $\Phi = \mathrm{Con}(u,v)$ in C, then $\Phi|A \subseteq \Theta$.

Proof: Let Ψ_A and Ψ_C be the smallest distributive congruences on A and C respectively, and let $\{u_s/v_s : s \in S\}$ be the set of all N–quotients of C. (i) ⟹ (ii): Assume (i). Then $\Psi_A = \Psi_C|A$ by Lemma 2.3. But by Lemma 0.3(a), $\Psi_C = \Sigma_S \mathrm{Con}(u_s/v_s)$ and so we see that (ii) holds. (ii) ⟹ (i): Assume (ii). Then $\Sigma_S(\mathrm{Con}(u_s/v_s)|A) = \Psi_A$ so that $\Psi_A = \Psi_C|A$ by Corollary 0.5 B(ii). Thus (i) holds by Lemma 2.3.

Note that the implication (i) ⟹ (ii) above holds in any variety $\mathcal{V}$ which does not contain the diamond.

Corollary 2.5: Let $\mathcal{V}$ be as in the previous Lemma. Let $A \leq C$ with $A \in \mathrm{Amal}(\mathcal{V})$ and $C \in \mathcal{V}$. Let $M \in \mathcal{V}_{MI}$ with critical quotient u/v and let $g : A \longrightarrow M$ be a homomorphism such that g[A] a distributive sublattice of M containing {u,v}. Suppose that $h : C \longrightarrow M$ is a homomorphism with $h|A = g$. Then $h[C] \subseteq (v] \cup [u)$ and so h[C] is an ordinal sum of two distributive lattices.
Proof: Since $A \in \mathrm{Amal}(\mathcal{V})$ it follows from Lemma 2.1 that every 2–congruence on A can be extended to a 2–congruence on C. Further, by Lemma 2.4 and Lemma 0.5 A(ii)(c) we see that h[C] must be a distributive sublattice of M. By Lemma 0.5 A(ii)(b), (v] and [u) form

distributive sublattices of M. The result follows easily now since $g[A] \leq h[C]$ and $g[A]$ contains $\{u,v\}$.

Our next lemma is concerned with restrictions of 2–congruences of a product onto its factors. The proof of the following lemma is the same as that of Lemma 1.2.5 in [3] but we include it here for completeness.

Lemma 2.6: Let $A = \Pi_I A_i$ be a product of lattices and assume that each for all $i \in I$, A_i has a 2–congruence. Let Θ be a 2–congruence A. Then there is an $j \in I$ and an embedding $g : A_j \longrightarrow A$ such that $\Theta|g[A_j]$ is a 2–congruence on $g[A_j]$.

Proof: There are $u,v \in A$ such that $A/\Theta = \{u/\Theta, v/\Theta\}$ and let $J = \{i \in I : u_i > v_i\}$, where u_i and v_i denote the i^{th} components of u and v respectively. There are two cases:

Case 1: There is a $j \in I$ such that $\{u_j,v_j\}$ is a 2–sublattice of A_j. Then there is a homomorphism $f : A_j \longrightarrow 2$ with $f(u_j) = 1$ and $f(v_j) = 0$. Define $g : A_j \longrightarrow A$ as follows: Let $x \in A_j$. Put $g(x)_j = x$. For $i \in J - \{j\}$ put $g(x)_i = u_i$ if $f(x) = 1$ and put $g(x)_i = v_i$ if $f(x) = 0$. For $i \in I - J$ put $g(x)_i = u_i = v_i$. Then g is an embedding with $g(u_j) = u$ and $g(v_j) = v$ and so we see that $\Theta|g[A_j]$ is a 2–congruence on $g[A_j]$.

Case 2: Not Case 1, i.e. for all $i \in I$ the set $\{u_i,v_i\}$ is not a 2–sublattice of A_i. Since Θ is completely meet irreducible in Con(A), by Jónsson's Lemma there is a filtral congruence Ψ on A such that $\Psi \subseteq \Theta$. Let $\mathcal{D}$ be an ultrafilter on I which induces Ψ. Then $J \in \mathcal{D}$. Also for all $i \in J$, $\{i\} \notin \mathcal{D}$ and so $J - \{i\} \in \mathcal{D}$. $J \neq \phi$ and so we may choose a $j \in I$. A_j has a 2–congruence and so there is a surjective homomorphism $f : A_j \longrightarrow 2$. Define $g : A_j \longrightarrow A$ as in Case 1. If $x \in A_j$ with $f(x) = 1$ then we see that $g(x)/\Theta = u/\Theta$ and if $y \in A_j$ with $f(y) = 0$ then $g(y)/\Theta = v/\Theta$. It follows that $\Theta|g[A_j]$ is a 2–congruence on $g[A_j]$.

For the next proposition let $A \in \mathrm{Amal}(\mathcal{B})$ and let Ω be a congruence on A such that A/Ω is a subdirect power of B. Let $\{\varphi_i : i \in S\}$ be the set of all B–congruences on A. Then there is

a subset $I \subseteq S$ and a regular subdirect representation $r : A/\Omega \longrightarrow B^I$ such that for all $i \in I$, $\varphi_i = \ker(\pi_i \circ r \circ k)$ where $\pi_i : B^I \longrightarrow B$ is the i^{th} projection and $k : A \longrightarrow A/\Omega$ is the canonical quotient map. Let $z \in B$ be fixed. Let $h : B^I \longrightarrow B^S$ be the embedding defined as follws: For $x \in B^I$, $h(x)_i = x_i$ for all $i \in I$ and $h(x)_i = z$ for all $i \in S - I$. Put $D = h[B^I]$ and put $C = h\circ r[A/\Omega]$. Then $C \leq D \leq B^S$. For $\Theta \in \mathrm{Amal}(\mathcal{B})$ with $\Omega \subseteq \Theta$ let $\bar{\Theta}$ denote the congruence on C corresponding to the congruence Θ/Ω on A/Ω under $h\circ r$.

To prove the following proposition we need the following result: Every member of $\mathrm{Amal}(\mathcal{B})$ is a subdirect power of B. This will be proved in section 4 as Lemma 4.1.

Proposition 2.7: Suppose $\Theta \in \mathrm{Con}(A)$ is such that $\bar{\Theta}$ is a 2–congruence on C. Then $\bar{\Theta}$ can be extended to a 2–congruence on D.

Proof: For each $i \in I$ let $\rho_i : A \longrightarrow B$ be the surjective homomorphism given by $\rho_i(x) = r(x/\Omega)_i$ for all $x \in A$. Then for all $i \in I$, $\varphi_i = \ker(\rho_i)$. For each $i \in S - I$ let $\rho_i : A \longrightarrow B$ be any surjective homomorphism with $\varphi_i = \ker(\rho_i)$. Define a map $\rho : A \longrightarrow B^S$ as follows: For $x \in A$, $\rho(x)_i = \rho_i(x)$ for all $i \in S$. By Lemma 4.1 below (which does not require Proposition 2.7 in its proof), A is a subdirect power of B and so we see that $\rho : A \longrightarrow B^S$ is in fact a subdirect representation. Since $\bar{\Theta}$ is a 2–congruence on C, Θ is a 2–congruence on A and so there are $u,v \in A$ with $u > v$ such that $A/\Theta = \{u/\Theta, v/\Theta\}$. Then $C/\bar{\Theta} = \{r(u/\Omega)/\bar{\Theta}, r(v/\Omega)/\bar{\Theta}\}$. By Lemma 2.1 there is a 2–congruence Γ on B^S such that the congruence $\Gamma|\rho[A]$ on $\rho[A]$ corresponds to Θ under ρ. By Jónsson's Lemma there is a filtral congruence Ψ on B^S with $\Psi \subseteq \Gamma$. Let $\mathcal{D}$ be a ultrafilter over S which induces Ψ. Since B is finite, Ψ is a B–congruence. Also $B^S/\Gamma = \{\rho(u)/\Gamma, \rho(v)/\Gamma\}$ and so there is a non–critical quotient p/q of B for which the set $X = \{i \in S : \rho_i(u) = p \text{ and } \rho_i(v) = q\}$ belongs to $\mathcal{D}$. There are two cases:

Case 1: $I \in \mathcal{D}$. Then $\rho(u)/\Gamma = r(u/\Omega)/\Gamma$ and $\rho(v)/\Gamma = r(v/\Omega)/\Gamma$ whence it follows that $\Gamma|D$ is a 2–congruence on D extending $\bar{\Theta}$.

Case 2: $I \notin \mathcal{D}$. Then $X - I \in \mathcal{D}$. Let $t : C \longrightarrow 2$ be a surjective homomorphism with $\ker(t) = \bar{\Theta}$. Define $f : C \longrightarrow B^S$ as follows: For $x \in C$ let $y \in A$ with $x = r(y/\Omega)$. For all $i \in I$ put $f(x)_i = \rho_i(y)$. For all $i \in S - (X \cup I)$ put $f(x)_i = z$. For all $i \in X - I$ put $f(x)_i = p$ if $t(x) = 1$ and put $f(x)_i = q$ if $t(x) = 0$. Then f is an embedding and the congruence $\Gamma | f[C]$ on $f[C]$ corresponds to the congruence $\bar{\Theta}$ on C under f. There exists a surjective homomorphism $w : B^I \longrightarrow 2$. Define $g : D \longrightarrow B^S$ as follows: For $x \in D$ put $g(x)_i = x_i$ for all $i \in I \cup (S - X)$. For all $i \in X - I$ put $g(x)_i = p$ if $w(x) = 1$ and put $g(x)_i = q$ if $w(x) = 0$. Then g is an embedding. Let Δ be the congruence on D corresponding to the congruence $\Gamma | g[D]$ on $g[D]$ under g. Then Δ is a 2–congruence on D. Now $D \cong B^I$ and so by Theorem 0.9A(ii), $D \in \mathrm{Amal}(\mathcal{B})$. Thus there is a $Q \in \mathcal{B}$ and embeddings $g', j : B^S \longrightarrow Q$ such that $g' \circ g = j | D$. Also by Theorem 0.9A(ii), $B^S \in \mathrm{Amal}(\mathcal{B})$ and so by Lemma 2.1 there is a congruence Λ on Q such that the congruence $\Lambda | g'[B^S]$ on $g'[B^S]$ corresponds to the congruence Γ on B^S under g'. Let Ξ be the congruence on D corresponding to the congruence $\Lambda | j[B^S]$ on $j[B^S]$ under j. Then we have $\Xi | D = \Delta$. Since the variety of all lattices satisfies the amalgamation property (see [8]) there is a lattice R and embeddings $f', e : B^S \longrightarrow R$ such that $f' \circ f = e | C$. Let Ξ' be the congruence on $f'[B^S]$ corresponding to the congruence Ξ on B^S under f'. Then $\Xi' | f[C] = \Gamma | f[C]$ and so $\Xi' | f[C]$ corresponds to $\bar{\Theta}$ under f. But $\Xi | C$ corresponds to $\Xi' | f[C]$ under f and so in fact $\Xi | C = \bar{\Theta}$. Since $\Xi | D = \Delta$ we then have $\Delta | C = \bar{\Theta}$ and so Δ is an extension of $\bar{\Theta}$ to a 2–congruence on D as required.

3. FINITE IMAGES OF MEMBERS OF AMAL($\mathcal{B}$)

Lemma 3.1: Let $A \in \mathrm{Amal}(\mathcal{B})$. Then:

(i) **3** is not an image of A.

(ii) Every finite distributive image of A is Boolean.

Proof: (i) Suppose that **3** is an image of A. There exists an embedding $f : A \longrightarrow B^I$ for some index set I. Also there is a homomorphism $g : A \longrightarrow B$ with $g[A] \cong \mathbf{3}$ and $a, c \in g[A]$ (where c/a is the unique critical quotient of B). By Theorem 0.9A(i) there is a homomorphism $h : B^I \longrightarrow B$ such that $h \circ f = g$. Since **3** is distributive it follows from Corollary 2.5 that $h[B^I]$ is an ordinal sum of distributive lattices. However, this leads to a contradiction since by Proposition 1.2.1 (ii) $h[B^I]$ must be a Boolean lattice with more than two elements. Thus **3** cannot be an image of A.

(ii) follows from (i).

The proof of the following result is the same as the proof of Lemma 2.4 in [3] but we include it for completeness.

Lemma 3.2: Let $C \in \mathcal{B}$ and let Θ_0 and Θ_1 be 2–congruences on C and put $\Phi = \Theta_0 \cap \Theta_1$. Let $A \in \mathrm{Amal}(\mathcal{B})$ and let $f_0, f_1 : A \longrightarrow C$ be two embeddings. Assume that $\Theta_0 | f_0[A]$ and $\Theta_1 | f_1[A]$ correspond, under f_0 and f_1 respectively, to two distinct congruences on A. Then Φ not a 3–congruence on C.

Proof: Suppose that Φ is a 3–congruence. Since **3** is a projective lattice there $u, v, z \in C$ with $u > v > w$ such that $C/\Theta_0 = \{u/\Theta_0, v/\Theta_0\}$, $C/\Theta_1 = \{v/\Theta_1, w/\Theta_1\}$, $\{u,v\} \subsetneq f_0[A]$ and $\{v,w\} \subsetneq f_1[A]$. Then $C/\Phi = \{u/\Phi, v/\Phi, z/\Phi\}$. The diagram (A, f_0, C, f_1, C) has an amalgam (D, g_0, g_1) in $\mathcal{B}$. Then by Lemma 2.1 there are 2–congruences Θ_0' and Θ_1' on D such that $\Theta_0' | g_0[C]$ corresponds to Θ_0 under g_0 and $\Theta_1' | g_1[C]$ corresponds to Θ_1 under g_1. Then $D/\Theta_0' = \{ g_0(u)/\Theta_0', g_0(v)/\Theta_0' \}$ and $D/\Theta_1' = \{ g_1(v)/\Theta_1', g_1(w)/\Theta_1' \}$. Let $x, y, z \in A$ such that $f_0(x) = u$, $f_0(y) = v$ and $f_1(z) = w$. Put $\Phi' = \Theta_0' \cap \Theta_1'$ and let $h = g_0 \circ f_0$ $(= g_1 \circ f_1)$. Then we see that Φ' is a 3–congruence on D with $D'/\Phi = \{ h(x)/\Phi', h(y)/\Phi', h(z)/\Phi' \}$. Let Γ be the congruence on A corresponding to $\Phi' | h[A]$ under h. Then Γ is a 3–congruence on A with $A/\Gamma = \{x/\Gamma, y/\Gamma, z/\Gamma\}$. But this contradicts Lemma 3.1(i) and so Φ is not a 3–congruence on C.

We shall now turn our attention to non–modular subdirectly irreducible images of members of Amal($\mathcal{B}$). It turns out that every such image is isomorphic to $\mathcal{B}$ (see Corollary 3.4 below.)

Lemma 3.3: Let $A \in$ Amal($\mathcal{B}$) and let X be a non–modular image of A with $X \leq B$. Then X is automorphic in B.

Proof: Let $g_1, g_2 : X \longrightarrow B$ be embeddings.. Let $k : A \longrightarrow X$ be a surjective homomorphism. By Lemma 2.7 we have a subdirect representation $f : A \longrightarrow B^I$. By Theorem 0.9 A(i) there are homomorphisms $h_1, h_2 : B^I \longrightarrow B$ with $h_1 \circ f = g_1 \circ k$ and $h_2 \circ f = g_2 \circ k$. Note that $h_1[B^I]$ and $h_2[B^I]$ must both contain c/a since $g_1[X]$ and $g_2[X]$ do by non–modularity. By Lemma 0.5 D, $h_1[B^I]$ and $h_2[B^I]$ are both images of B, and so it follows that they must be isomorphic to B. It follows that h_1 and h_2 are surjective. Thus there is an automorphism $t : B \longrightarrow B$ with $h_1 = t \circ h_2$. Then $t \circ g_1 \circ k = t \circ h_1 \circ f = h_2 \circ f = g_2 \circ k$ and so, since k is surjective, we have $t \circ g_1 = g_2$. Thus X is automorphic in B.

Corollary 3.4: Let $A \in$ Amal($\mathcal{B}$) and let X be a non–modular subdirectly irreducible image of A. Then X is isomorphic to B.

Proof: By Proposition 1.1.1(i), $X \in \mathcal{K}$. Thus by Corollary 1.1.6, X is either a B–lattice or X is isomorphic to L^n_5 for some non–negative integer n. The latter case is ruled out since **3** is not an image of A (see Lemma 3.1(i)). Thus X is a B–lattice. Since B is the only member of $\mathcal{B}_{MI}$ up to isomorphism, X is embeddable in B. By the preceding lemma X is automorphic in B. The result follows now from Corollary 1.2.3.

4. MEMBERS OF AMAL($\mathcal{B}$)

The principal result of this section is Theorem 4.2 which characterizes the members of Amal($\mathcal{B}$). This characterization is needed to prove Theorem 5.1.

Lemma 4.1: Every member of Amal($\mathcal{B}$) is a subdirect power of B.

Proof: Let $A \in$ Amal($\mathcal{B}$). By Corollary 0.9I, B is a subdirect product of non–modular subdirectly irreducible members of $\mathcal{B}$. The result follows from Corollary 3.4.

Theorem 4.2: Let $A \in \mathcal{B}$. Then the following are equivalent:

(a) $A \in$ Amal($\mathcal{B}$).

(b) **3** is not an image of A and for any $C \in \mathcal{B}$ with $A \leq C$, every 2–congruence on A can be extended to a 2–congruence on C.

(c) A is a subdirect power of B and for any regular subdirect representation $f : A \longrightarrow B^I$ and any homomorphism $g : A \longrightarrow B$, there is a homomorphism $h : B^I \longrightarrow B$ with $h \circ f = g$.

(d) A is a subdirect power of B and for any subdirect representation $f : A \longrightarrow B^I$ and any homomorphism $g : A \longrightarrow B$, there is a homomorphism $h : B^I \longrightarrow B$ with $h \circ f = g$.

(e) A is a subdirect power of B, **3** is not an image of A and if $f : A \longrightarrow B^I$ is a regular subdirect representation, then any 2–congruence on f[A] can be extended to a congruence on B^I.

Proof: (a) $\Longrightarrow$ (b) by Lemma 2.1 and Lemma 3.1(i). (b) $\Longrightarrow$ (a): Assume (b). We use Theorem 0.9A(i). Let $g : A \longrightarrow B$ be a homomorphism and let $f : A \longrightarrow C$ be an embedding with $C \in \mathcal{B}$. There are two cases:

Case 1: g[A] is a distributive sublattice of B. Since **3** is not an image of A, g[A] must be Boolean. By (b), every 2–congruence on f[A] can be extended to a 2–congruence on C and so by Lemma 2.2(ii) there is a homomorphism $h : C \longrightarrow B$ with $h \circ f = g$.

Case 2: g[A] is a non–modular sublattice of B. Then g[A] must contain the unique N–quotient c/a of B. It follows that g[A] is a B–lattice. Since every subdirectly irreducible member of $\mathcal{B}$ is embeddable in B there is an embedding $t : C \longrightarrow M$ where M is a subdirect power of B. Thus by Lemma 1.2.5 there is a homomorphism $h' : M \longrightarrow B$ with $h' \circ t \circ f = g$. Putting $h = h' \circ t : C \longrightarrow B$ we have $h \circ f = g$. Thus in $A \in$ Amal($\mathcal{B}$) by Theorem 0.9A(i).

(a) $\Longrightarrow$ (c) by Lemma 4.1 and Theorem 0.9A(i) while (c) $\Longrightarrow$ (a) by Lemma 0.9D. (a) $\Longleftrightarrow$ (d) by Lemma 4.1 and Lemma 0.9E. (a) $\Longrightarrow$ (e) by Lemma 2.1, Lemma 4.1 and Lemma 3.1(i). (e) $\Longrightarrow$ (a): Assume (e): We again use Theorem 0.9A(i). Let $g : A \longrightarrow B$ be a homomorphism and let $f : A \longrightarrow C$ with $C \in \mathcal{B}$. As in the proof of (b) $\Longrightarrow$ (a), $g[A]$ is either Boolean or a B–lattice. Using Lemma 2.2(ii) in the first case and Lemma 1.2.5 in the second case we obtain a homomorphism $h : C \longrightarrow B$ with $h \circ f = g$ as required.

The equivalence of (a) and (e) above generalizes Theorem 2.2 of [3]. The equivalence of (a) and (d) above generalizes Theorem 6.2 in [9]. The following theorem generalizes Theorem 6.1 in [9].

Theorem 4.3: Let $A \in \mathcal{B}$ be finite. Then $A \in \mathrm{Amal}(\mathcal{B})$ if and only if A is a subdirect power of B and **3** is not an image of A.

Proof: By the equivalence of (a) and (e) in Theorem 4.2 it suffices to show that if $f : A \longrightarrow B^I$ is a regular subdirect representation, then every 2–congruence on $f[A]$ can be extended to a 2–congruence on B^I. So let Θ be a 2–congruence on $f[A]$. By Jónsson's Lemma there is a filtral congruence Ψ on B^I such that $\Delta \subseteq \Theta$ where $\Delta = \Psi | f[A]$. Note that since A is finite and f is regular, I is finite and so Ψ is induced by a principal ultrafilter. It follows that there is an $i \in I$ such that $\Psi = \ker(\pi_i)$ where $\pi_i : B^I \longrightarrow B$ is the i^{th} projection. Since f is a subdirect representation it follows that there is an isomorphism $t : B \longrightarrow f[A]/\Delta$ given by the the mapping $x \longmapsto \{y \in f[A] : y_i = x\}$. Let Γ be the congruence on B corresponding to the congruence Θ/Δ on $f[A]/\Delta$ under t. Define a relation Θ' on B^I by $(x,y) \in \Theta'$ if and only if $(x_i,y_i) \in \Gamma$. Then Θ' is a 2–congruence on B^I extending Θ.

We are now going to consider certain extensions of members of $\mathrm{Amal}(\mathcal{B})$ which also belong to $\mathrm{Amal}(\mathcal{B})$. The proof of Proposition 4.4 below is the same as that of Proposition 2.5 in [3] but we include it for completeness.

Proposition 4.4: Let $C \in \mathcal{B}$ and assume that for any two distinct 2–congruences Θ_0 and Θ_1 on C there is an $A \in \mathrm{Amal}(\mathcal{B})$ and embeddings $f_0,f_1 : A \longrightarrow C$ such that $\Theta_0 | f_0[A]$ and $\Theta_1 | f_1[A]$ correspond, under f_0 and f_1 respectively, to two distinct congruences on A. Then $C \in \mathrm{Amal}(\mathcal{B})$.

Proof: If $D \in \mathcal{B}$ with $C \leq D$ then it is easily seen that any 2–congruence on C can be extended to a 2–congruence on D. Also, by Lemma 3.2, C cannot have **3** as an image. The result now follows from the equivalence of (a) and (b) in Theorem 4.2.

5. THE MAIN THEOREM

Theorem 5.1: $\mathrm{Amal}(\mathcal{B})$ is an elementary class. It is closed under reduced products and is therefore definable by Horn sentences. Furthermore, if C is both an image of a member of $\mathrm{Amal}(\mathcal{B})$ and a subdirect power of B, then $C \in \mathrm{Amal}(\mathcal{B})$.

Proof: We first prove the last statement of the theorem. Let $f : C \longrightarrow B^I$ be a regular subdirect representation and suppose that C is an image of some $A \in \mathrm{Amal}(\mathcal{B})$. Using Proposition 2.8 we see that any 2–congruence on $f[C]$ can be extended to a 2–congruence on B^I. Since $A \in \mathrm{Amal}(\mathcal{B})$, **3** is not an image of A and hence not of C. By the equivalence of (a) and (e) in Theorem 4.1, $C \in \mathrm{Amal}(\mathcal{B})$. We now show that $\mathrm{Amal}(\mathcal{B})$ is closed under reduced products. Let $C = \Pi_{\mathcal{F}} A_i$ be a reduced product where $\mathcal{F}$ is a filter over a set I and $A_i \in \mathrm{Amal}(\mathcal{B})$ for all $i \in I$. Let $A = \Pi_I A_i$. We first show that $A \in \mathrm{Amal}(\mathcal{B})$, using Proposition 4.3. So suppose we have two distinct 2–congruences Θ_0 and Θ_1 on A. We show that there is a $j \in I$ and embeddings $f_0,f_1 : A_j \longrightarrow A$ such that $\Theta_0 | f_0[A_j]$ and $\Theta_1 | f_1[A_j]$ correspond, under f_0 and f_1 respectively, to two distinct congruences on A_j. Now $A/(\Theta_0 \cap \Theta_1)$ is isomorphic to either **3** or $\mathbf{2} \times \mathbf{2}$. In either case we have $u,v,w \in A$ with $u > v > w$ such that $(u,v) \notin \Theta_1$, $(v,w) \in \Theta_1$, $(v,w) \notin \Theta_0$ and $(u,v) \in \Theta_0$. By Jónsson's Lemma there are filtral congruences Ψ_0 and Ψ_1 on A with $\Psi_0 \subseteq \Theta_0$ and $\Psi_1 \subseteq \Theta_1$. Let $\mathcal{D}_0$ and $\mathcal{D}_1$ be ultrafilters on I that induce Ψ_0 and Ψ_1 respectively. Put $R = \{i \in I : u_i > v_i\}$ and put

$S = \{i \in I : v_i > w_i\}$. Then $R \in \mathcal{D}_0$ and $S \in \mathcal{D}_1$. For all $i \in I$ let $z_i \in A_i$ be fixed. There are three possible cases:

(1) There is a $j \in I$ such that $\{j\} \in \mathcal{D}_0$ and $\{j\} \in \mathcal{D}_1$.

(2) For all $i \in I$, $\{i\} \notin \mathcal{D}_0$ and $\{i\} \notin \mathcal{D}_1$.

(3) There is a $j \in I$ such that $\{j\} \in \mathcal{D}_0$ but $\{j\} \notin \mathcal{D}_1$, or vice versa.

For (1): Define $f_0 : A_j \longrightarrow A$ as follows: For all $x \in A_j$, put $f_0(x)_j = x$ and put $f_0(x)_i = z_i$ for all $i \in I - \{j\}$. Put $f_1 = f_0$. For (2): Since we are considering a case where A has 2–congruences there is a $j \in I$ such that A_j is non–trivial. Then $R - \{j\} \in \mathcal{D}_0$ and $S - \{j\} \in \mathcal{D}_1$. A_j is a non–trivial subdirect power of B and so there are two distinct surjective homomorphisms $k,t : A_j \longrightarrow 2$. Define $f_0 : A_j \longrightarrow A$ as follows. For all $x \in A_j$: Put $f_0(x)_j = x$. For all $i \in R - \{j\}$, put $f_0(x)_i = u_i$ if $k(x) = 1$ and put $f_0(x)_i = v_i$ if $k(x) = 0$. For all $i \in I - (R \cup \{j\})$ put $f_0(x)_i = z_i$. Define $f_1 : A_j \longrightarrow A$ as follows. For all $x \in A_j$: Put $f_1(x)_j = x$. For all $i \in S - \{j\}$ put $f_1(x)_i = v_i$ if $t(x) = 1$ and put $f_1(x)_i = w_i$ if $t(x) = 0$. For all $i \in I - (S \cup \{j\})$ put $f_1(x)_i = z_i$. For (3): If there is a $j \in I$ with $\{j\} \in \mathcal{D}_0$ but $\{j\} \notin \mathcal{D}_1$, then we define f_0 as in (1) and f_1 as in (2) while if there is a $j \in I$ with $\{j\} \in \mathcal{D}_1$ but $\{j\} \notin \mathcal{D}_0$ we define f_1 as in (1) and f_0 as in (2). In all three cases we see that $\Theta_0|f_0[A_j]$ and $\Theta_1|f_1[A_j]$ correspond, under f_0 and f_1 respectively, to two distinct congruences on A_j. Thus by Proposition 4.4, $A \in \mathrm{Amal}(\mathcal{B})$. By Proposition 0.6, C is a subdirect product of non–modular subdirectly irreducible members of $\mathcal{B}$. Now C is an image of A. Since $A \in \mathrm{Amal}(\mathcal{B})$, every non–modular subdirectly irreducible image of C must be isomorphic to B (see Corollary 3.4). Thus C is a subdirect power of B. By the last statemant of the theorem which we have already proved, $C \in \mathrm{Amal}(\mathcal{B})$ as required. Since $\mathrm{Amal}(\mathcal{B})$ is closed under reduced products it is closed under ultraproducts in particular. However any amalgamation class of a variety which is closed under ultraproducts is elementary (see [13]). Lastly, any elementary class closed under reduced products is definable by Horn sentences (see [4]).

Note that the above theorem generalizes Theorem 3.1 in [3].

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Department of Mathematics
University of Cape Town
Rondebosch 7700
South Africa

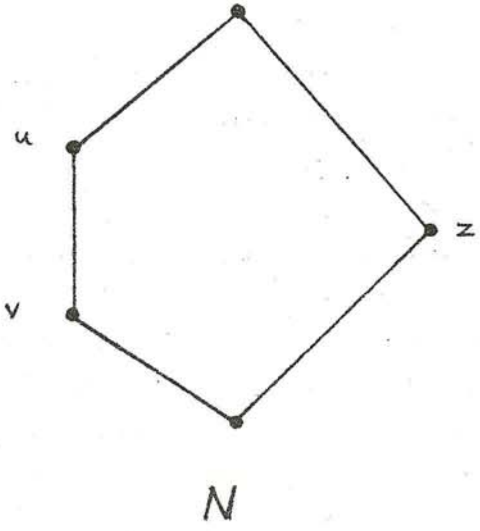


Figure 1

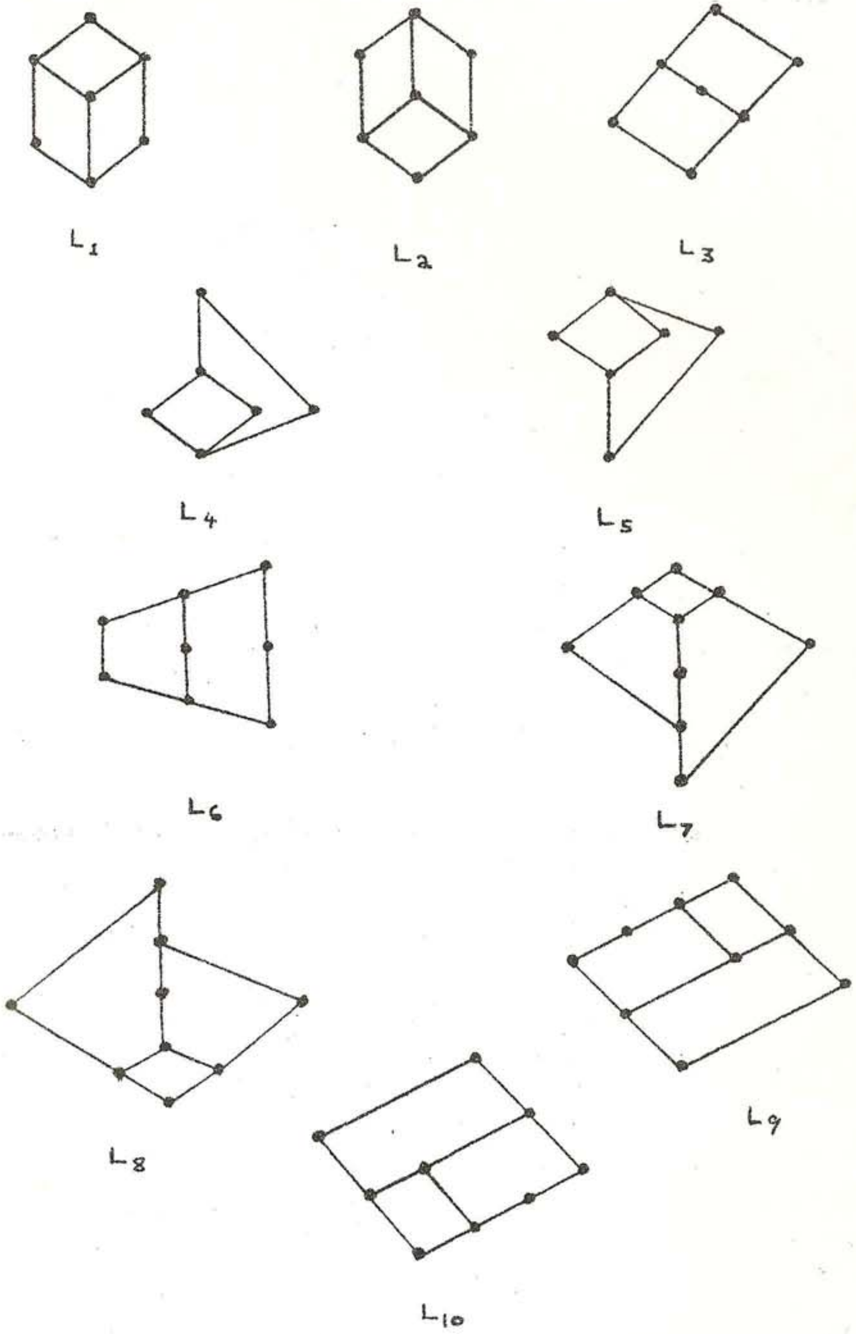
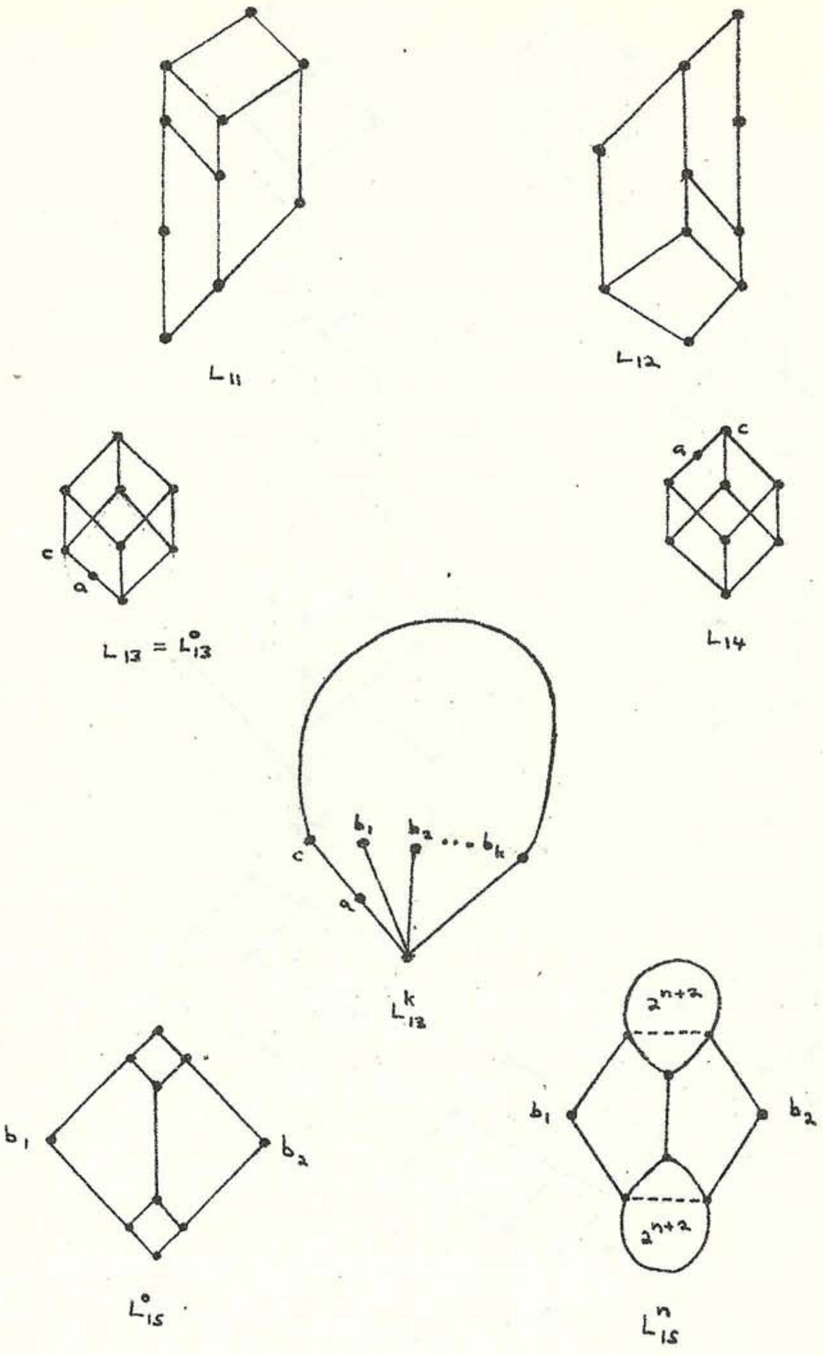


Figure 2



L_{11}

L_{12}

$L_{13} = L^{0}_{13}$

L_{14}

L^{k}_{13}

L^{0}_{15}

L^{n}_{15}

Figure 3